Real Analysis II - 6CCM321A

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Chapter 1

Metrics and Norms

Definition 1.1. *Let X be a non-empty set. A function $\rho : X \times X \to \mathbb{R}$ is called a **metric** on X if it satisfies*

- $\rho(x, y) > 0$ *if* $x \neq y$ *with equality if and only if* $x = y$
- $\rho(x, y) = \rho(y, x)$
- $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$

where $x, y, z \in X$.

Example 1.2. *The function* $\rho(x, y) = |x - y|$ *is a metric on both* $\mathbb{R}$ *and* $\mathbb{C}$.

Example 1.3. *Consider* $\mathbb{R}^n$ *and* $\mathbb{C}^n$. *Then the function*

$$\rho(x, y) = \left(\sum_{i=1}^{n} |x_i - y_i|^2 \right)^{\frac{1}{2}}$$

is a metric for these spaces. Here x_i *and* y_i *are the* i^{th} *coordinates of the points* x *and* y *respectively.*

Example 1.4. *Consider* $X = C[a, b]$, *the set of all continuous functions on* $[a, b]$. *Then the function*

$$\rho(f, g) = \sup_{x \in [a,b]} |f(x) - g(x)|$$

is a metric on X.

Example 1.5. *Consider $X = B(S)$, the set of all bounded functions on a set S. Then the function*

$$\rho(f, g) = \sup_{x \in S} |f(x) - g(x)|$$

is a metric on X.

Example 1.6. *Consider an arbitrary set X. Then the function*

$$\rho(x, y) = \begin{cases} 1 & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases}$$

*is a metric on X called the **discrete metric**.*

Definition 1.7. *If X is a non-empty set and ρ a metric on X, we define the pair (X, ρ) as a **metric space**.*

Definition 1.8. *Let (X, ρ) be a metric space and $A \subseteq X$ a subset. The metric space (A, ρ) is called a **subspace** of (X, ρ).*

Example 1.9. *$C[a, b]$ is a subspace of $B[a, b]$ with their usual metric for $-\infty < a < b < \infty$. Indeed, any continuous function on a bounded closed interval is always bounded.*

Definition 1.10. *Let X be a vector space over $\mathbb{R}$ (or $\mathbb{C}$). A function $|| \cdot || : X \to \mathbb{R}$ is called a **norm** if it satisfies the following axioms*

- $||x|| > 0$ *if* $x \neq 0$ *and* $||0|| = 0$
- $||\lambda x|| = |\lambda| \, ||x| \; \forall \, x \in X, \forall \, \lambda \in \mathbb{R}$ *(or* $\lambda \in \mathbb{C}$*)*
- $||x + y|| \leq ||x|| + ||y|| \; \forall \, x, y \in X$

*A linear space equipped with a norm is called a **normed space**.*

Proposition 1.11. *Let $|| \cdot ||$ be a norm on X. Then the function $\rho(x, y) = ||x - y||$ is a metric on X.*

Proof.

1. We first have to show that $\rho(x, y) \geq 0$ with equality if and only if $x = y$. Since $|| \cdot ||$ is a norm on X, we have that $\rho(x, y) = ||x - y|| > 0$. We also have that $||0|| = 0 \iff ||x - x|| = 0$. We see that $\rho(x, y) = 0 \iff x = y$.

2. We now have to show that $\rho(x, y) = \rho(y, x)$. We have that $\rho(x, y) = ||x - y||$. Now since $||\cdot||$ is a norm, we know that $||\lambda x|| = |\lambda| \, ||x|| \; \forall \, \lambda \in \mathbb{R}$. Choosing $\lambda = -1$, we see that $\rho(x, y) = ||x - y|| = ||y - x|| = \rho(y, x)$.

3. Finally, we show that $\forall \, x, y, z \in X, \rho(x, z) \leq \rho(x, y) + \rho(y, z)$. We have that $\rho(x, z) = ||x - z|| = ||x - y + y - z|| = ||(x - y) - (z - y)|| \leq ||x - y|| + ||z - y|| = ||x - y|| + ||y - z|| = \rho(x, y) + \rho(y, z)$.

□

Example 1.12. *The metric in Example 1.2 is generated by the norm* $||x|| = x$. *The metric in Example 1.3 is generated by* $||x|| = (\sum_{i=1}^{n} |x_i|^2)^{\frac{1}{2}}$.

Example 1.13. *The metrics in Examples 1.4 and 1.5 are generated by*

$$||f|| = \sup_{x \, \in \, [a,b]} |f(x)|$$

$$||f|| = \sup_{x \, \in \, S} ||f(x)||$$

Chapter 2

Convergence

Definition 2.1. *Consider a sequence of elements x_n of a metric space (X, ρ). x_n is said to* ***converge*** *to $x \in X$ if*

$$\forall \varepsilon > 0 \, \exists n_\varepsilon \in \mathbb{N} \textit{ such that } \forall n \geq n_\varepsilon, \ \rho(x_n, x) \leq \varepsilon$$

Lemma 2.2. *Let (X, ρ) be a metric space and x_n a sequence of elements in X. Then $x_n \to x$ in (X, ρ) if and only if $\rho(x_n, x) \to 0$ in $\mathbb{R}$.*

Proof. This follows directly from the definition of convergent sequences in a metric space and on the real line. □

Lemma 2.3. *Consider a sequence of elements x_n in (X, ρ). If r_n are non-negative numbers such that $r_n \to 0$ in $\mathbb{R}$ and $\rho(x_n, x) \leq r_n$ for all sufficiently large n then $x_n \to x$ in (X, ρ).*

Proof. By the definition of convergence on the real line, we have that

$$\forall \varepsilon > 0, \ \exists n_\varepsilon \in \mathbb{N} \text{ such that } \forall n \geq n_\varepsilon, r_n \leq \varepsilon$$

Now fix $\varepsilon > 0$ and choose n_ε satisfying above. We see that $\rho(x_n, x) \leq r_n \leq \varepsilon \ \forall n \geq n_\varepsilon$. This implies that $\rho(x_n, x) \to 0$ in $\mathbb{R}$ and thus by the previous lemma, $x_n \to x$ in (X, ρ). □

Definition 2.4. *Let ρ_1 and ρ_2 be two metrics defined on a set X. We say that the ρ_1 and ρ_2 are* ***equivalent*** *when*

$$x_n \to x \textit{ in } (X, \rho_1) \iff x_n \to x \textit{ in } (X, \rho_2)$$

Definition 2.5. *We define **uniform convergence on S** to be convergence in the metric space $B(S)$.*

Lemma 2.6. *Let f_n be a sequence of functions in $B(S)$ and $f \in B(S)$. Then*

$$f_n \to f \iff \forall\, \varepsilon > 0\, \exists\, n_\varepsilon \in \mathbb{N} \text{ such that } \forall\, n \geq n_\varepsilon, \forall\, x \in S, |f_n(x) - f(x)| \leq \varepsilon$$

Proof. First we note that $f_n \to f$ if and only if $\rho(f_n, f) = \sup_{x\in S} |f_n(x) - f(x)| \to 0$. By definition, we have that the sequence of real numbers $sup_{x\in S}|f_n(x) - f(x)|$ converges to 0 if and only if

$$\forall\, \varepsilon > 0\, \exists\, n_\varepsilon \in \mathbb{N} \text{ such that } \forall\, n \geq n_\varepsilon, sup_{x\in S}|f_n(x) - f(x)| \leq \varepsilon$$

Hence we just have to show that $sup_{x\in S}|f_n(x) - f(x)| \leq \varepsilon$ if and only if $|f_n(x) - f(x)| \leq \varepsilon\ \forall\, x \in S$.

$\Longrightarrow$:
Now since $sup_{x\in S}|f_n(x) - f(x)| \leq \varepsilon$, it follows that $|f_n(x) - f(x)| \leq \sup_{x\in S} |f_n(x) - f(x)| \leq \varepsilon\ \forall\, x \in S$.

$\Longleftarrow$:
Now assume that $\forall\, \varepsilon > 0\ \exists\, n_\varepsilon \in \mathbb{N}$ such that $\forall\, n \geq n_\varepsilon, \forall\, x \in S, |f_n(x) - f(x)| \leq \varepsilon$. Since the supremum of a set coincides with the least upper bound, we can always find a $\delta > 0$ and a point $x_\delta \in S$ such that

$$\sup_{x\in S} |f(x) - f_n(x)| \leq \delta + |f(x_\delta) - f_n(x_\delta)| \leq \delta + \varepsilon$$

We can choose δ to be arbitrarily small and in the limit $\delta \to 0$, we have that $\sup_{x\in S} |f(x) - f_n(x)| \leq \varepsilon$.

□

Remark. *A sequence of functions $f_n \in B(S)$ converges to $f \in B(S)$ **pointwise** if $\forall\, x \in S, \forall\, \varepsilon > 0, \exists\, n_{\varepsilon,x} \in \mathbb{Z}$ such that $|f(x) - f_n(x)| \leq \varepsilon$ for all $n > n_{\varepsilon,x}$. Such an integer may depend on x. If for any ε, the set $\{n_{\varepsilon,x}\}_{x\in S}$ is bounded above then $f_n \to f$ uniformly.*

Chapter 3

Open and Closed Sets

For the following definitions and results, let (X, ρ) be a metric space and $r > 0$ some real constant.

Definition 3.1. *Let $\alpha \in X$. Then the set*

$$B_r(\alpha) = \{\, x \in X \mid \rho(x, \alpha) < r \,\}$$

*is called an **open ball** of radius r centered at α. The set*

$$B_r[\alpha] = \{\, x \in X \mid \rho(x, \alpha) \leq r \,\}$$

*is called a **closed ball** of radius r centered at α.*

Definition 3.2. *Let $\alpha \in X$ and $A \subseteq X$ a subset. We say that A is a **neighbourhood** of α if there exists an open ball $B_r(\alpha) \subseteq A$ for some $r > 0$.*

Remark. *We can reformulate the definition of convergence of a sequence using open balls as follows. A sequence x_n in a metric space converges to a point x if for any ball $B_\varepsilon(x)$, there exists an integer n_ε such that for all $n > n_\varepsilon$ then $x_n \in B_\varepsilon(x)$.*

Theorem 3.3. *Consider two metrics ρ and σ on the same set X. Then ρ and σ are equivalent if and only if every open ball $B_r^\rho(x)$ contains an open ball $B_s^\sigma(x)$ and every open ball $B_s^\sigma(x)$ contains an open ball $B_t^\rho(x)$.*

Proof.

$\implies$: Assume that the two metrics are equivalent. Furthermore, assume there exists an open ball $B_r^\rho(x)$ which does not contain an open ball $B_s^\sigma(x)$

with $s > 0$. Consider a sequence $s_n \to 0$ and let $x_n \in B^\sigma_{s_n}(x)$ and $x_n \notin B^\rho_r(x)$. Then obviously, $x_n \to x$ in the metric σ. However, x_n does not converge to x in the metrox ρ because $x_n \notin B^\rho_r(x)$ for any n. Hence the metrics ρ and σ are not equivalent. But this contradicts our assumption.

$\Longleftarrow$: Now assume that every open ball $B^\rho_r(x)$ contains an open ball $B^\sigma_s(x)$ and every open ball $B^\sigma_s(x)$ contains an open ball $B^\rho_t(x)$. Given any sequence $x_n \overset{\sigma}{\to} x$ then for any $r > 0$, we can choose an $s > 0$ and a corresponding n_s such that

$$x_n \in B^\sigma_s(x) \subseteq B^\rho_r(x)$$

for all $n > n_s$. Hence $x_n \overset{\rho}{\to} x$. Applying the same argumentation to a ρ-ball contained in a σ-ball implies given any sequence $x_n \overset{\rho}{x}$ then also $x_n \overset{\sigma}{x}$.

□

3.1 Open sets

Definition 3.4. *Let $A \subseteq X$ be a subset. We say that A is **open** if it contains a ball about each of its points.*

Lemma 3.5. *An open ball in a metric space (X, ρ) is open.*

Proof. Let $x \in B_r(\alpha)$. Then $\rho(x, \alpha) = r - \varepsilon$ for some $\varepsilon > 0$. Consider another point $y \in B_\varepsilon(x)$. Then $\rho(y, x) < \varepsilon$ and by the triangle inequality we have that

$$\rho(y, \alpha) \leq \rho(y, x) + \rho(x, \alpha) < \varepsilon + r - \varepsilon = r$$

Therefore, $y \in B_r(\alpha)$ for all $y \in B_\varepsilon(x)$. Hence, $B_\varepsilon(x) \subseteq B_r(\alpha)$. □

Theorem 3.6. *Let (X, ρ) be a metric space. Then we have that*

1. *X and $\varnothing$ are both open*
2. *the union of any collection of open subsets of X is open*
3. *the intersection of any finite collection of open subsets of X is open*

Proof.

Part 1: The whole space is obviously open as it contains all open balls. The empty space contains no points and is therefore trivially open.

Part 2: Let $x \in \cup_n A_n$ for some open sets A_n. Then $x \in A_i$ for some i. Since A_i is open, it contains an open ball around x. Obviously this ball is also contained in A_n and thus A_n is open.

Part 3: Let $A_1, \ldots, A_k$ be open sets and $x \in \cup_{n=1}^{k} A_n$. Then $x \in A_n$ for every n. Since each A_n are open, for each n there exist an $r_n > 0$ such that $B_{r_n}(x) \subseteq A_n$. Now let $r = \min\{r_1, \ldots, r_n\}$. Then $r > 0$ and $B_r(x) \subseteq B_{r_n}(x) \subseteq A_n$ for all $n = 1, \ldots, k$. Hence $B_r(X) \subseteq \cap_{n=1}^{k} A_n$.

□

Lemma 3.7. *A set is open if and only if it is the union of a collection of open balls.*

Proof. $\Longrightarrow$: Let A be an open set. We have to show that it is the union of a collection of open balls. Let $x \in A$. Since A is open, there exists an open ball $B(x)$ around x. Then obviously $A = \cup_{x \in A} B(x)$.

$\Longleftarrow$: Now assume that A is the union of a collection of open balls. We know that open balls are open sets and also arbitrary unions of open sets are again open. Therefore A must be open.

□

Definition 3.8. *Let $x \in A$. We say that x is an* ***interior*** *point of A if there exists an open ball $B_r(x) \subseteq A$ for some $r > 0$. We define the* ***interior*** *of a set A to be the union of all open sets contained in A. In other words, it is the maximal open set contained in A. The interior of A is denoted by $int(A)$.*

3.2 Closed sets

Definition 3.9. *Let X be a set and $A \subseteq X$ a subset. We say that a point $x \in X$ is a* ***limit point*** *of the set A if every ball about x contains a point of A distinct from x. We denote the set of limit points of A by A'.*

Lemma 3.10. *Let A be a set. Then x is a limit point of A if and only if there is a sequence x_n of elements of A distinct from x which converges to x.*

Proof.

$\Longrightarrow$: Let x be a limit point of A. By the definition of a limit point, we know that every open ball around x contains a point of A distinct from x. Consider the ball $B_{\frac{1}{n}}(x)$. By the previous statement, such a ball will contain a point of a sequence $x_n \in A$ which is distinct from x. Since this ball converges to x, obviously $x_n \to x$.

$\Longleftarrow$: Now assume that there exists a sequence x_n of elements of A distinct from x that converges to x. It follows trivially that any ball around x must contain an element of x_n.

□

Definition 3.11. *We say that a set is **closed** if it contains all of its limit points.*

Lemma 3.12. *A set A is closed if and only if the limit of any convergent sequence of elements of A lies in A.*

Proof. $\Longrightarrow$: Assume that A is closed. By definition, it contains all of its limit points. Since the limit of a convergent sequence either coincides with an element of A or one of its limit points, all convergent sequences must converge to a point in A.

$\Longleftarrow$: Now assume that the limit of any convergent sequence of elements of A lies in A. By definition, a limit point of A is a limit of some sequence $\{x_n\} \subseteq A$. It follows that A must contain all of its limit points and thus is closed.

□

Lemma 3.13. *Let (X, ρ) be a metrix space. Then any closed ρ-ball in X is a closed set.*

Proof. Let x_n be a convergent sequence lying in the closed ball $B_r[\alpha]$ with a limit x. By the triangle inequality, we have that

$$\rho(x, \alpha) \leq \rho(x, x_n) + \rho(x_n, \alpha) \leq \rho(x, x_n) + r$$

Since $x_n \to x$, $\rho(x, x_n) \to 0$ for large n. This implies that $\rho(x, \alpha) \leq r$ and thus $x \in B_r[\alpha]$. Hence $B_r[\alpha]$ contains all of its limit points and it must be a closed set. □

Definition 3.14. *Let X be a set and $A \subseteq X$ a subset. Then we define the **complement** $\mathcal{C}(A)$ of A to be all points $x \in X$ which do not belong to A.*

Theorem 3.15. *If A is open then $\mathcal{C}(A)$ is closed. If A is closed then $\mathcal{C}(A)$ is open.*

Proof. Let A be open. Then for every point of A there exists an open ball around the point, contained in A. Clearly such a ball does not contain any points from $\mathcal{C}(A)$. Therefore no point of A can be a limit point of $\mathcal{A}$. Indeed, if there did exist such a limit point then there would exist an open ball around the point containing some points of $\mathcal{C}(A)$ distinct from the point. We must therefore have that $\mathcal{C}(A)$ contains all of its limit points and is therefore closed.
Now assume that A is closed. Then, by definition, A contains all of its limit points. Therefore $\mathcal{C}(A)$ cannot contain a limit point of A. Therefore for any $x \in \mathcal{C}(A)$, there exists a ball $B_r(X)$ which is contained in $\mathcal{C}(A)$. Thus $\mathcal{C}(A)$ is open. □

Theorem 3.16. *Let (X, ρ) be a metric space. Then*

1. *X and $\varnothing$ are closed*
2. *arbitrary intersections of closed sets are closed*
3. *the union of any finite collection of closed sets is closed*

Proof. The proof is left as an exercise to the reader. It follows from Theorem 3.6, Theorem 3.15 and the application of De Morgan's Law. □

Definition 3.17. *Let A be a set. We define the **closure**, denoted $\overline{A}$, of A to be the intersection of all closed sets containing A. In other words, it is the minimal closed set containing A.*

Theorem 3.18. *Let A be a set. Then $\overline{A} = A \cup A'$.*

Proof. We first show that $A \cup A'$ is a closed set. Let $x \in \mathcal{C}(A \cup A')$. Obviously $x \notin A'$. Therefore there exists a ball $B_r(x)$ which does not contain elements of A distinct from x. Since also $x \notin A$, we have that $B_r(x) \subseteq \mathcal{C}(A)$. Since $B_r(x)$ is an open ball, it must be an open set. Hence for any $y \in B_r(x)$, there exists a ball $B_\varepsilon(y) \subseteq B_r(x) \subseteq \mathcal{C}(A)$ that also does not contain any points of A distinct from x. Hence $B_r(x) \subseteq \mathcal{C}(A \cup A')$. We see that we can excise

an open ball around any point of $\mathcal{C}(A \cup A')$ and therefore $\mathcal{C}(A \cup A')$ is open. This implies that $A \cup A'$ is closed.
It now suffices to show that $A \cup A'$ is the minimal closed set containing A. Let K be a closed set such that $A \subseteq K \subseteq A \cup A'$ and $K \neq A \cup A'$. Obviously K must not contain at least one limit point of A'. Therefore there must be a sequence of elements $x_n \in A \subseteq K$ which converges to this point. But we know that a set is closed if and only if it contains the limit point of any of its convergent sequences. Therefore K is not closed - a contradiction. □

Corollary 3.19. *Let A be a set. Then $x \in \overline{A}$ if and only if there exists a sequence $\{x_n\} \subseteq A$ which converges to x.*

Proof.

$\Longrightarrow$: Assume that $x \in \overline{A}$. Then either $x \in A$ or $x \in A'$. Assume that $x \in A$. Then obviously the sequence $\{x, x, \ldots, x\} \subseteq A$ converges to x. Now assume that $x \in A'$. Lemma 3.10 implies that there must exist a sequence x_n of elements of A converging to x.

$\Longleftarrow$: Now assume that there exists a sequence $\{x_n\}$ consisting of elemments of A which coverges to x. If $x_n = x$ for some n then $x \in A$. If not then Lemma 3.11 implies that $x \in A'$. Therefore $x \in \overline{A} = A \cup A'$.

□

Example 3.20. *Let (X, ρ) be a metric space equipped with the discrete metric*

$$\rho(x,y) = \begin{cases} 1 & \textit{if } x \neq y \\ 0 & \textit{if } x = y \end{cases}$$

Then the ρ-balls are given by

$$B_r[a] = \begin{cases} \{a\} & \textit{if } r < 1 \\ X & \textit{if } r \geq 1 \end{cases}, B_r(a) = \begin{cases} \{a\} & \textit{if } r \leq 1 \\ X & \textit{if } r > 1 \end{cases}$$

Since the open ball is open, any point is an open set. Since the union of open sets is open, we have that every subset of X is open. It therefore follows that every subset of X is also closed (as it is they are complements of open sets).

Theorem 3.21. *In a normed linear space, $\overline{B_r(\alpha)} = B_r[\alpha]$.*

Proof.

$\overline{B_r(\alpha)} \subseteq B_r[\alpha]$: Obviously, $B_r(\alpha) \subseteq B_r[a]$. Now the closure of a set is the minimal closed set containing that set. We also know that closed balls are closed sets. Hence we have that $\overline{B_r(\alpha)} \subseteq B_r[\alpha]$.

$\overline{B_r(\alpha)} \supseteq B_r[\alpha]$: We need to show that $B_r[\alpha] \subseteq \overline{B_r(\alpha)} = B_r(\alpha) \cup (B_r(\alpha))'$. Let $x \in B_r(\alpha)$. Then $\rho(\alpha, x) \leq r$. If $\rho(\alpha, x) < r$ then $x \in B_r(\alpha)$ and we are done. Hence assume that $\rho(\alpha, x) = r$ and consider $x_n = x_n^{-1}(\alpha - x)$. We have that

$$\begin{aligned}
\rho(x_n, \alpha) &= ||x_n - \alpha|| \\
&= ||x - \alpha + n^{-1}(\alpha - x)|| \\
&= (1 - n^{-1})||x_n - \alpha|| \\
&= (1 - n^{-1})r \\
&< r
\end{aligned}$$

This shows that $x_n \in B_r(\alpha)$ for all n. Now,

$$\begin{aligned}
\rho(x_n, x) &= ||x_n - x|| \\
&= ||n^{-1}(\alpha - x)|| \\
&= n^{-1}||x - \alpha|| \\
&= n^{-1}r
\end{aligned}$$

We see that in the limit $n \to \infty$, $\rho(x_n, x) \to 0$. Therefore x is a limit point of the set $B_r(\alpha)$ and $x \in (B_r(\alpha))'$.

□

Chapter 4

Continuity

Definition 4.1. *Let (X, ρ) and (Y, d) be metric spaces. A map $T : X \to Y$ is said to be **continuous at** $\alpha \in X$ if*

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ such that whenever } \rho(x, \alpha) < \delta \text{ then } d(Tx, T\alpha) < \varepsilon$$

*The map T is said to be **continuous** if it is continuous at every point $\alpha \in X$.*

Theorem 4.2. *Let (X, ρ) and (Y, d) be metric spaces and $T : X \to Y$ a mapping. Then T is continuous at $\alpha \in X$ if and only if for every sequence x_n converging to α in (X, ρ), the sequence Tx_n converges to $T\alpha$ in (Y, d).*

Proof. $\implies$: First assume that T is continuous at $\alpha \in X$ and $x_n \to \alpha$ in (X, ρ). We need to show that for all $\varepsilon > 0$, there exists an n_ε such that for all $n > n_\varepsilon$, $d(Tx_n, T\alpha) \leq \varepsilon$.
Now fix $\varepsilon > 0$ since T is continuous at α, we can always find a $\delta > 0$ such that $\rho(x_n, \alpha) < \delta \implies d(Tx_n, T\alpha)$. Since the sequence x_n converges to α in (X, ρ) for this δ there exists an n_δ such that $\rho(x_n, \alpha)\delta$. Obviously, taking $n_\varepsilon := n_\delta$ satisfies the condition.

$\impliedby$: Now assume that for any sequence x_n that converges to α in (X, ρ), the sequence Tx_n converges to $T\alpha$ in (Y, d). Suppose, for a contradiction, that T is not continuous. Then there exists an $\varepsilon_0 > 0$ such that for any $\delta > 0$, there is an $x \in X$ for which $\rho(x, \alpha) < \delta$ and $d(Tx, T\alpha) \geq \varepsilon_0$. We can choose $x_n \in X$ such that $\rho(x_n, \alpha) < \frac{1}{n}$ and $d(Tx_n, T\alpha) \geq \varepsilon_0$. Then $x_n \to \alpha$ in (X, ρ) but Tx_n does not converge to $T\alpha$ in (Y, d) - a contradiction.

$\square$

Theorem 4.3. *Let $(X,\rho),(Y,d)$ and (Z,σ) be metric spaces. Consider a continuous at α mapping $T_1 : (X,\rho) \to (Y,d)$ and a continuous at $T_1\alpha$ mapping $T_2 : (Y,d) \to (Z,\sigma)$. Then $T_2T_1 : (X,\rho) \to (Z,\sigma)$ is continuous at α.*

Proof. Let x_n be a sequence converging to α in (X,ρ). Since T_1 is continuous at α, $T_1x_n \to T_1\alpha$ in (Y,d). Now since T_2 is continuous at $T_1\alpha, T_2T_1x_n = T_2(T_1x_n) \to T_2(T_1\alpha)$ in (Z,σ). Applying Theorem 4.3, we see that T_2T_1 is continuous. □

Lemma 4.4. *Let (X,p) be a metric space and $x_0 \in X$ a fixed element. Then $T : x \to \rho(x,x_0)$ is a continuous map from (X,ρ) to $\mathbb{R}$.*

Proof. We need to show that given a sequence $x_n \to \alpha$ in (X,ρ), $Tx_n \to T\alpha$ in $\mathbb{R}$. We have that

$$\begin{aligned}
|Tx_n - T\alpha| &= |\rho(x_n,x_0) - \rho(\alpha,x_0)| \\
&= |\rho(x_n,x_0) + \rho(x_n,\alpha) - \rho(x_n,\alpha) - \rho(\alpha,x_0)| \\
&\leq |\rho(x_n,\alpha)| + |\rho(x_n,x_0) - \rho(x_n,\alpha) - \rho(\alpha,x_0)| \\
&\leq \rho(x_n,\alpha) + |\rho(x_n,\alpha) + \rho(\alpha,x_0) - \rho(x_n,\alpha) - \rho(\alpha,x_0)| \\
&\leq \rho(x_n,\alpha)
\end{aligned}$$

This implies that $Tx_n \to T\alpha$ whenever $x_n \to \alpha$. □

Definition 4.5. *Let (X,ρ) and (Y,d) be metric spaces. Then we can define a metric on their Cartesian product $X \times Y$*

$$\sigma((x_1,y_1),(x_2,y_2)) = \sqrt{(\rho(x_1,x_2))^2 + (d(y_1,y_2))^2}$$

Remark. *We note that given the continuity of a function of two variables with the above metric is not the same as continuity in each variable seperately. For example*

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } x^2+y^2 \neq 0 \\ 0 & \text{if } x=y=0 \end{cases}$$

defined on $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$ is continuous at the origin in each variable seperately but discontinuous as a function of two variables.

Definition 4.6. *Given a mapping $T : X \to Y$ and a subset $A \subseteq Y$. The set*

$$T^{-1}(A) = \{ x \in X \mid Tx \in A \}$$

is called the ***inverse image*** *of A.*

Definition 4.7. *(Alternate formulation of continuity)*
Let (X, ρ) and (Y, d) be metric spaces. A mapping $T : X \to Y$ is said to be ***continuous*** *at $\alpha \in X$ if for any open ball $B_\varepsilon(T\alpha)$ about $T\alpha$, there exists a ball $B_\delta(\alpha)$ about α such that $B_\delta(\alpha) \subseteq T^{-1}(B_\varepsilon(T\alpha))$.*

Theorem 4.8. *Let (X, ρ) and (Y, σ) be metric spaces and $T : X \to Y$ a mapping from $X \to Y$. Then the following are equivalent:*

1. *T is continuous*
2. *the inverse image of every open subset of Y is an open subset of X*
3. *the inverse image of every closed subset of Y is a closed subset of X*

Proof. We shall prove the theorem in the order $(2) \iff (3), (1) \iff (2)$.

$(2) \iff (3)$: The inverse image of the complement of a set A is exactly the complement of the inverse image $T^{-1}(A)$ hence the equivalence of (2) and (3) follows from Theorem 3.15.

$(1) \implies (2)$: Assume that T is continuous and let A be an open subset of Y and $x \in T^{-1}(A) \subseteq X$. Since A is open, there exists an open ball $B_\varepsilon(Tx)$ about the point Tx such that $B_\varepsilon(Tx) \subseteq A$. Now since T is a continuous map, we have that there exists an open ball $B_\delta(x) \subseteq T^{-1}(B_\varepsilon(Tx)) \subseteq T^{-1}(A)$. Therefore for every point we can excise an open ball around any $x \in T^{-1}(A)$. It thus follows that $T^{-1}(A)$ is an open set.

$(2) \implies (1)$: Now assume that the inverse image of any open set is open. Let $x \in X$ and $B_\varepsilon(Tx)$ be a ball around $T_x \in Y$. The inverse image $T^{-1}(B_\varepsilon(Tx))$ is an open set containing the point x. Therefore there is an open ball $B_\delta(x)$ about x such that $B_\delta(x) \subseteq T^{-1}(B_\varepsilon(Tx))$. Hence T is continuous.

□

Definition 4.9. *Let X and Y be normed linear spaces equipped with the norms $||\cdot||_X$ and $||\cdot||_Y$ respectively. We say that a linear map $T : X \to Y$ is* ***bounded*** *if there exists a positive constant C such that $||T_x||_Y \leq C||x||_X$ for all $x \in X$.*

Theorem 4.10. *Let (X, ρ) and (Y, d) be normed spaces and $T : X \to Y$ a linear map. Then the following are equivalent:*

1. *T is continuous*
2. *T is continuous at 0*
3. *T is bounded*

Proof. (1) $\implies$ (2) is satisfied trivially. We shall prove the theorem in the order (2) $\implies$ (3) $\implies$ (1).

(2) $\implies$ (3): Let T be continuous at 0. Then there exists a $\delta > 0$ such that $\rho(0, z) = ||z|| \leq \delta \implies d(0, Tz) = ||Tz|| \leq 1$. For arbitrary $x \neq 0$ in X, the element $z = \delta ||x||^{-1} x$ satisfies the condition $||z|| = \delta$ implying that $||Tz|| \leq 1$.
Now since T is a linear map, we have that

$$||Tx|| = ||T(z\delta^{-1}||x||)|| = \delta^{-1}||x|| \, ||Tz|| \leq d^{-1}||x||$$

Obviously this inequality is valid for $x = 0$ as well. Therefore T is bounded with $C = \delta^{-1}$.

(3) $\implies$ (1): Now assume that T is bounded. Then $||T_x||_Y \leq C||x||X$ for all $x \in X$. Fix $\varepsilon > 0$ and take $z \in X$. Furthermore, set $\delta = C^{-1}\varepsilon$. Then for all $x \in X$ such that $\rho(x, z) = ||x - z||_X < \delta$, we have that

$$d(Tx, Tz) = ||Tx - Tz||_Y = ||T(x - z)||_Y \leq C||x - z||_X < \varepsilon$$

Hence T is continuous.

∎

Chapter 5

Completeness

5.1 Cauchy Sequences

Definition 5.1. *Let (X, ρ) be a metric space. We say that a sequence of elements $x_n \in X$ is a* ***Cauchy sequence*** *if*

$$\forall\, \varepsilon > 0, \exists\, n_\varepsilon \text{ such that } \forall\, n > n_\varepsilon, \rho(x_n, x_m) < \varepsilon$$

Lemma 5.2. *Let (X, ρ) be a metric space. Then every convergent sequence in X is a Cauchy sequence.*

Proof. Let $x_n \to x$ be a convergent sequence. Then for any $\varepsilon > 0$, there exists an n_ε such that $\rho(x_n, x) < \frac{\varepsilon}{2}$ for all $n > n_\varepsilon$. By the triangle inequality, it follows that

$$\rho(x_n, x_m) \leq \rho(x_n, x) + \rho(x_m, x) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

for all $n, m > n_\varepsilon$. Hence x_n is a Cauchy sequence. □

Lemma 5.3. *Let (X, ρ) be a metric space. If a Cauchy sequence in X has a convergent subsequence then it is convergent to the same limit.*

Proof. Let x_n be a Cauchy sequence and $x_{n_k} \to x$ a convergent subsequence. By definition we have that for any $\varepsilon > 0$, there exists an n_ε such that $\rho(x_n, x_{n_k}) < \frac{\varepsilon}{2}$ and $\rho(x_{n_k}, x) < \frac{\varepsilon}{2}$ for all $n, n_k > n_\varepsilon$. By the triangle inequality, it follows that

$$\rho(x_n, x) \leq \rho(x_{n_k}, x) + \rho(x_n, x_{n_k}) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

for all $n > n_\varepsilon$. Therefore $x_n \to x$. □

5.2 Complete Metric Spaces

Definition 5.4. *Let (X, ρ) be a metric space. We say that (X, ρ) is **complete** if any Cauchy sequence $\{x_n\} \subseteq X$ converges to a limit $x \in X$.*

Theorem 5.5. *Let (X, ρ) be a metric space. Then there exists a complete metric space $(\tilde{X}, \tilde{\rho})$ such that*

1. *$X \subseteq \tilde{X}$ and $\tilde{\rho}(x, y) = \rho(x, y)$ whenever $x, y \in X$*
2. *for all $\tilde{x} \in \tilde{X}$ there exists a sequence of elements $x_n \in X$ such that $x_n \to \tilde{x}$ in the limit $n \to \infty$ in the space $(\tilde{X}, \tilde{\rho})$*

*We say that the metric space $(\tilde{X}, \tilde{\rho})$ is the **completion** of (X, ρ).*

Theorem 5.6. *Let (A, ρ) be a subspace of a complete metric space (X, ρ). Then $(\overline{A}, \rho)$ is the completion of (A, ρ).*

Proof. Consider a Cauchy sequence $\{x_n\} \subseteq \overline{A}$. Now since (X, ρ) is complete, we are guaranteed that that $x_n \to x$ for some $x \in X$. But $\overline{A}$ is closed and must contain all its limit points. Therefore $x \in \overline{A}$. Hence $(\overline{A}, \rho)$ is complete. By Corollary 3.19, we know that an element of $\overline{A}$ must be the limit point of a sequence $\{s_n\} \subseteq A$. Therefore $(\overline{A}, \rho)$ myst be the completion of (A, ρ). □

Example 5.7. *Let (X, ρ) be the rational numbers equipped with the standard metric $\rho(x, y) = |x - y|$. Consider a sequence that converges to an irrational number such as $(3, 3.1, 3.14, 3.141, \dots) \to \pi$. Obviously such a sequence does not have a limit point in the rational numbers. Therefore X is not complete. We define the real numbers $\mathbb{R}$ equipped with the metric ρ to be the completion of X.*

Example 5.8. *The complex numbers $\mathbb{C}$ equipped with the standard metric $\rho(x, y) = |x - y|$ are complete. Indeed consider a sequence $\{c_n\} \subseteq \mathbb{C}$. Let $c_n = a_n + ib_n$ for some $a_n, b_n \in \mathbb{R}$. Then $\{c_n\}$ is a Cauchy sequence if and only if the sequences $\{a_n\}$ and $\{b_n\}$ are Cauchy sequences of real numbers. We also have that the sequence $\{c_n\}$ converges if and only if the sequences $\{a_n\}$ and $\{b_n\}$ converge.*

Theorem 5.9. *Consider the set $B(S)$ equipped with the metric $\rho(f, g) = \sup_{x \in S} |f(x) - g(x)|$. Then $B(S)$ is complete.*

Proof. We have to show that given a Cauchy sequence $\{f_n\} \subseteq B(S)$, there exists an $f \in B(S)$ such that $f_n \to f$ uniformly. By definition of a Cauchy sequence, we have that

$$\forall \varepsilon > 0 \, \exists \, n_\varepsilon \text{ such that } \forall \, n, m > n_\varepsilon, \sup_{x \in S} |f_n(x) - f_m(x)| < \frac{\varepsilon}{2}$$

Now fix some $x \in S$. Obviously, in light of the above, the numbers $f_n(x)$ form a Cauchy sequence of real (or complex) numbers. We know that the space of real (or complex) numbers is complete and therefore, for a fixed x, the sequence $f_n(x)$ converges to some limit which we denote $f(x)$. In other words

$$\forall \, x \in S, \forall \, \varepsilon > 0, \exists \, n_{\varepsilon,x} \text{ such that } \forall \, n > n_{\varepsilon,x}, |f_n(x) - f(x)| < \frac{\varepsilon}{2}$$

Now we note that

$$\begin{aligned} |f_n(x) - f(x)| &= |f_n(x) + f_m(x) - f_m(x) - f(x) \\ &= |f_n(x) - f_m(x) + (f_m(x) - f(x))| \\ &\leq |f_n(x) - f_m(x)| - |f_m(x) - f(x)| \end{aligned}$$

Now choose n and m such that $n > n_\varepsilon$, $m > n_\varepsilon$ and $m > n_{\varepsilon,x}$. Then it follows that

$$\begin{aligned} |f_n(x) - f(x)| &\leq |f_n(x) - f_m(x)| - |f_m(x) - f(x)| \\ &= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

for all $x \in S$. Therefore the left hand side is less than or equal to ε for all $x \in S$ provided that $n > n_\varepsilon$. Since this n_ε is independent of x, we have that $f_n(x) \to f$ uniformly.
It remains to show that $f(x)$ is a bounded function. Indeed, choose $n > n_\varepsilon$ and

$$\begin{aligned} \sup_{x \in S} |f(x)| &= \sup_{x \in S} |f_n(x) - f_n(x) + f(x)| \\ &\leq \sup_{x \in S} (|f_n(x)| + |f(x) - f_n(x)|) \\ &\leq \sup_{x \in S} |f_n(x)| + \sup_{x \in S} |f(x) - f_n(x)| \\ &\leq \sup_{x \in S} |f_n(x)| + \varepsilon \end{aligned}$$

Now, by assumption, $f_n(x)$ is bounded and therefore $f(x)$ is also bounded. □

Corollary 5.10. *Consider $C[a,b]$ equipped with the metric $\rho(f,g) = \sup_{x\in[a,b]} |f(x) - g(x)|$. Then $C[a,b]$ is complete.*

Proof. Since continuous functions on closed intervals $[a,b]$ are bounded, we know that any sequence f_n of continuous must converge uniformly to a function f that is bounded on $[a,b]$. It suffices to show that f is continuous. In other words, we have to show that

$$\forall\, \varepsilon > 0\; \exists\, \delta > 0 \text{ such that if } |x-y| < \delta \text{ then } |f(x) - f(y)| < \varepsilon$$

Indeed, we have that

$$\begin{aligned}
|f(x) - f(y)| &= |f(x) + f_n(x) - f_n(x) + f_n(y) - f_n(y) - f(y)| \\
&= |f(x) - f_n(x) + (f_n(x) - f_n(y)) + (f_n(y) - f(y))| \\
&\leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)|
\end{aligned}$$

Now since $f_n \to f$ uniformly, we can choose an n_ε such that for all $n > n_\varepsilon$, $|f(x) - f_n(x)| < \frac{\varepsilon}{3}$ and $|f(y) - f_n(y)| < \frac{\varepsilon}{3}$. Furthermore, by assumption, we have that each f_n is a continuous function. Therefore we can always find a δ such that if $|x - y| < \delta$ then $|f(x) - f(y)| < \frac{\varepsilon}{3}$. Hence

$$\begin{aligned}
|f(x) - f(y)| &\leq |f(x) - f_n(x)| + |f_n(x) - f_n(y)| + |f_n(y) - f(y)| \\
&\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} \\
&= \varepsilon
\end{aligned}$$

whenever $|x - y| < \delta$. □

5.3 Series in Banach Spaces

Definition 5.11. *Let X be a normed linear space. Then X is called a **Banach space**.*

Example 5.12. *Theorem 5.9 and Corollary 5.10 imply that $B(S)$ and $C[a,b]$ are Banach spaces.*

For the following definitions and results, let x_n be a sequence of elements in a Banach space X.

Definition 5.13. *Consider the series $\sum_{n=1}^{\infty}$. We say that this series is **convergent** if the sequence $\sigma_k = \sum_{n=1}^{k} x_n$ is convergent in X. If $\sigma_k \to x \in X$ as $k \to \infty$, we write $\sum_{n=1}^{\infty} x_n = x$.*

Definition 5.14. *Let $\sigma = \sum_{n=1}^{\infty} x_n$ be a series. We say that σ is **absolutely** convergent if $\sum_{n=1}^{\infty} ||x_n|| < \infty$.*

Theorem 5.15. *Every absolutely convergent series in X is convergent in X.*

Proof. Let $\sigma_k = \sum_{n=1}^{k} x_n$ and $s_k = \sum_{n=1}^{k} ||x_n||$. Since the series $\sum_{n=1}^{\infty}$ is convergent, the sequence of positive numbers $\{s_k\}$ must also be convergent and it is therefore a Cauchy sequence. We therefore have that

$$\begin{aligned}
\rho(\sigma_m, \sigma_k) &= ||\sigma_m - \sigma_k|| \\
&= \left|\left| \sum_{n=k+1}^{m} x_n \right|\right| \\
&\leq \sum_{n=k+1}^{m} ||x_n|| \\
&= |s_m - s_k|
\end{aligned}$$

Now,

$$\forall\, \varepsilon > 0\, \exists\, n_\varepsilon \text{ such that } \forall\, m, k > n_\varepsilon, |s_m - s_k| < \varepsilon$$

Then let $m, k > n_\varepsilon$. It follows that

$$\begin{aligned}
\rho(\sigma_m, \sigma_k) &\leq |s_m - s_k| \\
&< \varepsilon
\end{aligned}$$

and thus $\{\sigma_n\}$ is a Cauchy sequence in the Banach space and thus it must converge. □

Corollary 5.16. *Let $\{f_n\} \subseteq B(S)$ be a sequence of bounded functions. If $\sum_{n=1}^{\infty} \sup |f_n(x)| < \infty$ then there exists a bounded function $f \in B(S)$ such that*

$$\sup_{x \in S} \left| f(x) - \sum_{n=1}^{k} f_n(x) \right| \overset{k \to \infty}{\to} 0$$

Furthermore, if $S = [a, b]$ is a bounded interval and the functions f_n are continuous on $[a, b]$ then f is also continuous.

Proof. Since $B(S)$ and $C[a, b]$ (with their usual norms) are Banach spaces, the corollary follows from Theorem 5.15. □

5.4 Contractions

Definition 5.17. *Let (X, ρ) be a metric space and $T : X \to X$ a mapping. We say that T is a* ***contraction*** *if $\rho(Tx, Ty) \leq c\rho(x, y)$ for some $0 \leq c < 1$ and for all $x, y \in X$.*

Example 5.18. *Consider $\mathbb{R}$ equipped with the standard metric $\rho(x, y)$. Then function*

$$\begin{aligned} f : \mathbb{R} &\to \mathbb{R} \\ x &\mapsto \frac{x}{2} \end{aligned}$$

is a contraction. Indeed, we have that

$$\rho(f(x), f(y)) = \left| \frac{x}{2} - \frac{y}{2} \right| = \frac{1}{2} |x - y| = \frac{1}{2} \rho(x, y)$$

Theorem 5.19. *(Contraction Mapping Theorem)*
Let T be a contraction on a complete metric space (X, ρ). Then the equation $Tx = x$ has a unique solution. Furthermore, for any $x_0 \in X$, the sequence $x_n = T^n x_0$ converges to x.

Proof. Let $n > m$ and fix $x_0 \in X$. Since T is a contraction we know that $\rho(Tx, Ty) \leq c\rho(x, y)$ for some $0 \leq< c < 1$. We have that

$$\begin{aligned} \rho(x_m, x_n) &= \rho(T^m x_0, T^n x_0) \\ &\leq c\rho(T^{m-1} x_0, T^{n-1} x_0) \\ &\leq c^2 \rho(T^{m-2} x_0, T^{n-2} x_0) \\ &\vdots \\ &\leq c^m \rho(x_0, T^{n-m} x_0) \end{aligned}$$

Now the triangle inequality implies that

$$\begin{aligned}\rho(x_0, T^{n-m}x_0) &\leq \rho(x_0, Tx_0) + \rho(Tx_0, T^{n-m}x_0)\\ &\leq \rho(x_0, Tx_0) + \rho(Tx_0, T^2x_0) + \rho(T^2x_0, T^{n-m}x_0)\\ &\leq \rho(x_0, Tx_0) + \rho(Tx_0, T^2x_0) + \rho(T^2x_0, T^3x_0) + \rho(T^3x_0, T^{n-m}x_0)\\ &\vdots\\ &\leq \rho(x_0, Tx_0) + \rho(Tx_0, T^2x_0) + \cdots + \rho(T^{n-m-1}x_0, T^{n-m}x_0)\\ &\leq \rho(x_0, Tx_0) + c\rho(x_0, Tx_0) + \ldots c^{n-m-1}\rho(x_0, Tx_0)\\ &= (1 + c + c^2 + \cdots + c^{n-m-1})\rho(x_0, Tx_0)\end{aligned}$$

We now note that $1 + c^2 + \cdots + c^{n-m-1} = \frac{1-c^{n-m}}{1-c}$. Since $0 \leq c < 1$ then obviously $0 \leq c^{n-m} < 1$. It thus follows that $1+c^2+\cdots+c^{n-m-1} \leq (1-c)^{-1}$. Combining the above results, we see that

$$\begin{aligned}\rho(x_m, x_n) &\leq c^m\rho(x_0, T^{n-m}x_0)\\ &\leq c^m(1 + c + c^2 + \cdots + c^{n-m-1})\rho(x_0, Tx_0)\\ &\leq c^m(1-c)^{-1}\rho(x_0, Tx_0)\end{aligned}$$

for all $x_0 \in X$ and for all $n > m$. Obviously the right hand side can be made arbitrarily small by choosing large m. This implies that $\{x_n\}$ is a Cauchy sequence. Furthermore, since the space is complete, $\{x_n\}$ converges to a limit $x \in X$. By Lemma 4.4, we know that the function $f(x) = \rho(x, Tx)$ is a continuous function. Therefore

$$\begin{aligned}\rho(x, Tx) &= \lim_{n\to\infty} \rho(x_n, Tx)\\ &= \lim_{n\to\infty} \rho(T^n x_0, Tx)\\ &\leq c \lim_{n\to\infty} \rho(T^{n-1}x_0, x)\\ &= c \lim_{n\to\infty} \rho(x_{n-1}, x)\\ &= 0\end{aligned}$$

Thus $\rho(x, Tx) = 0$ and hence $Tx = x$. Now let y be another solution such that $Ty = y$. Then

$$\begin{aligned}0 &= \rho(Tx, Ty) - \rho(x, y)\\ &\leq c\rho(x, y) - \rho(x, y)\\ &= (c-1)\rho(x, y)\end{aligned}$$

hence $\rho(x, y) = 0$. We therefore see that x is the only solution of the equation $Tx = x$. □

Remark. *The previous theorem allows us to construct an approximate solution to an equation of the form $Tx = x$. We choose an aribitrary element $x_0 \in X$ and evaluate $x_m = T^m x_0$ for sufficiently large m. This is called the* ***method of sucessive approximations****.*

Corollary 5.20. *The error for the method of successive approximations is given by*

$$\rho(x_m, x) \leq c^m (1-c)^{-1} \rho(x_0, Tx_0)$$

for all $x_0 \in X$ and $m \geq 0$.

Proof. This follows easily from the previous theorem and passing to the limit as $n \to \infty$. □

Example 5.21. *Let f be a real-valued function defined on an interval $[a, b]$ such that $f(x) \in [a, b]$ and*

$$|f(x) - f(y)| \leq c|x - y| \tag{5.1}$$

for all $x, y \in [a, b]$ and some constant $c < 1$. Then for any $x_0 \in [a, b]$, the sequence $x_1 = f(x_0), x_2 = f(x_1), x_3 = f(x_2), \ldots$ converges to the only solution of the equation $f(x) = x$.

Remark. *Inequality (5.3) is referred to as the* ***Lipschitz condition****. If f is continuously differentiable on $[a, b]$ then by the mean value theorem, f satisfies the Lipschitz condition with $c = \sup_{x \in [a,b]} |f'(x)|$.*

Chapter 6

Connectedness

Definition 6.1. *Let X be a metric space and $A \subseteq X$. We say that A is* ***connected*** *if there does not exist two non-empty open sets $U_1, U_2 \subseteq X$ such that $U_1 \cap U_2 = \varnothing$ and $A \subseteq U_1 \cup U_2$. If such a pair U_1, U_2 does exist, we say that U_1 and U_2 are a* ***disconnection*** *of the* ***disconnected*** *set A.*

Definition 6.2. *Let (X, ρ) be a metric space and let $x_1, x_2 \in X$. A continuous function $f : [0,1] \to X$ such that $f(0) = x_1$ and $f(1) = x_2$ is called a* ***path*** *from x_1 to x_2. A subset $A \subseteq X$ is said to be* ***path-connected*** *if there exists a path that lies entirely within A between any two points of A.*

Example 6.3. *A connected set is not necessarily path-connected. An example is the topologists sine curve:*

$$\left\{ \left(x, sin\frac{1}{x}\right) \,\middle|\, x \in (0,1] \right\} \cup \{ (0,0) \} \subseteq \mathbb{R}^2$$

This set is connected but not path-connected as the point $(0,0)$ cannot be reached by a non-constant path lying in the set.

Theorem 6.4. *A path-connected set is necessarily connected.*

Proof. Let A be a path-connected set and assume that A is not connected. In particular, let U_1 and U_2 form a disconnection of A. We have that $A \subseteq U_1 \cup U_2$ and $U_1 \cap U_2 = \varnothing$. Choose $x_1 \in U_1 \cap A$ and $x_2 \in U_2 \cap A$. Furthermore, let f be a path in A from x_1 to x_2. By definition, f is a continuous map and therefore the inverse images $f^{-1}(U_1)$ and $f^{-1}(U_2)$ are open subsets of $[0,1]$.

Obviously, these sets are non empty and

$$f^{-1}(U_1) \cap f^{-1}(U_2) = \varnothing$$
$$f^{-1}(U_1) \cup f^{-1}(U_2) = [0, 1]$$

In other words, $f^{-1}(U_1)$ and $f^{-1}(U_2)$ form a disconnection of $[0, 1]$.
Now let $r_1 := \sup f^{-1}(U_1)$. Since $0 \in f^{-1}(U_1)$ and $f^{-1}(U_1)$ is open in $[0, 1]$, there exists a positive number $r < 1$ such that the open ball $B_r(0) = [0, r)$ lies in $f^{-1}(U_1)$. Therefore $0 < r \leq r_1$. Similarly, since $1 \in f^{-1}(U_2)$ and $f^{-1}(U_2)$ is open, there exists $r < 1$ such that $B_r(1) = (r, 1]$ lies in $f^{-1}(U_2)$. Hence no point of $f^{-1}(U_1)$ lies in $(r, 1]$ and therefore $r_1 \leq r < 1$.
Now, the point r_1 cannot belong to $f^{-1}(U_1)$. Indeed if it did then there would exist an $\varepsilon > 0$ such that $(r_1 - \varepsilon, r_1 + \varepsilon) \subseteq f^{-1}(U_1)$, contradicting the fact that r_1 is the supremum. However, every neighbourhood of r_1 must contain a point from $f^{-1}(U_1)$. Hence if $r_1 \in f^{-1}(U_2)$, there would exist an open neighbourhood contained in $f^{-1}(U_2)$ which contains points of $f^{-1}(U_1)$, contradicting the fact that they are disconnected. But then $r_1 \notin f^{-1}(U_1) \cup f^{-1}(U_2)$ which is a contradiction. □

Lemma 6.5. *Any interval of $\mathbb{R}$ is path-connected.*

Proof. Let A be an interval and $x_1, x_2 \in A$ such that $x_1 < x_2$. Then $[x_1, x_2] \subseteq A$. Now define $f(t) = (1-t)x_1 + tx_2$ for $0 \leq t \leq 1$. Obviously, $f(t) \in [x_1, x_2]$ hence $f(t)$ is a path between x_1 and x_2 lying in A. We thus see that A is path connected. □

Lemma 6.6. *Any ball in a Banach space (over $\mathbb{R}$ or $\mathbb{C}$) is path connected.*

Proof. Let A be an open (or closed) ball and choose $x_1 \in A$. Then $f(t) = (1-t)x_1 + tx$ for $0 \leq t \leq 1$ is a path from x_1 to x. We have that

$$||f(t) - x|| = ||(1-t)x_1 + tx - x|| = (1-t)||x_1 - x|| \leq ||x_1 - x||$$

for all $t \in [0, 1]$. Therefore $f(t)$ is contained in A. Hence given any ball, we can find a path between any of its points and its centre. Since the ball is arbitrary and can be centered anywhere with any positive radius, the whole space must be path connected. □

Theorem 6.7. *Let X be a Banach space. Then every open connected set in X is path-connected.*

Proof. Let A be an open and connected set and fix $x \in A$. We define

$$U_1 = \{\, y \in A \mid \text{ there is a path from } x \text{ to } y \;\}$$

It can easily be seen that U_1 is path connected. Indeed, given any two points $y, z \in A$, let f_1 be the path from x to y and f_2 the path from x to z. Then

$$f(t) = \begin{cases} f_1(-2t) & \text{if } 0 \leq t \leq \frac{1}{2} \\ f_2(2t-1) & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases}$$

is a path from y to z. If U_1 coincides with A then A is path-connected and we are done. Hence assume that U_1 does not coincide with A. Consider $U_2 = \mathcal{C}(U_1) \cap A$. Since A is an open set, we can excise an open ball $B_r(y)$ around any point $y \in A$.

Now, $B_r(y) \cap U_1$ is empty. Indeed if it contains a point z then $z \in U_1$ and there exists a path between x and z. Since z is also in $B_r(y)$ and all balls in a Banach space are path connected, there must also exist a path between y and z. But then we can construct a path between x and y which would imply that $y \in U_1$ which is absurd. Hence $B_r(y) \subseteq U_2$. But y is an arbitrary point and hence U_2 is open. We also see that $A = U_1 \cup U_2$. But then U_1 and U_2 are a disconnection of A which contradicts that A is connected. Therefore $U_2 = \varnothing$ and $A = U_1$ and it is path-connected. □

Theorem 6.8. *Every connected subset of* $\mathbb{R}$ *is necessarily an interval.*

Proof. Let $A \subseteq \mathbb{R}$ be a connected subset of the real line. Furtermore, let $a = \inf A$ and $b = \sup A$ (if A is unbounded then we allow $a = -\infty$ and/or $b = \infty$). We claim that A necessarily contains every point between a and b. Fix such a point x. Since $a < x < b$, there must exist some point $\alpha \in A$ such that $\alpha < x$ (otherwise x would be a lower bound greater than a which is absurd). Similarly, there exists a $\beta in A$ such that $x < \beta$. Now suppose that $x \in A$. Then $(-\infty, x)$ (which contains α) and (x, ∞) form a disconnection of A. This is a contradiction to the fact that A is a connected subset. Therefore $x \in A$. Since x is an arbitrary point, A must contain all points between a and b and is therefore an interval. □

Theorem 6.9. *Let* (X, ρ) *and* (Y, σ) *be metric spaces and* $f : X \to Y$ *a continuous mapping. Then the image of any connected subset* A *of* X *under* f *is a connected subset of* Y.

Proof. Let $A \subseteq X$ be a connected subset and denote $B = f(A) \subseteq Y$. We need to show that B is a connected subset of Y. Suppose that B is disconnected. In particular, there exist two open sets $U_1, U_2 \subseteq Y$ such that $U_1 \cap U_2 = \varnothing$ and $B \subseteq U_1 \cup U_2$. We see that since U_1 and U_2 are open sets, $f^{-1}(U_1)$ and $f^{-1}(U_2)$ are also open since the inverse image of open sets under a continuous map is always open. Since $U_1 \cap U_2 = \varnothing$, we must have that $f^{-1}(U_1 \cap U_2) = f^{-1}(\varnothing)$. This implies that $f^{-1}(U_1) \cap f^{-1}(U_2) = \varnothing$. Furthermore, $f^{-1}(B) \subseteq f^{-1}(U_1 \cup U_2) = f^{-1}(U_1) \cup f^{-1}(U_2)$. Therefore $f^{-1}(U_1)$ and $f^{-1}(U_2)$ form a disconnection of A. But this contradicts the fact that A is connected. Therefore B must be connected. □

Example 6.10. *Let X be a metric space equipped with the discrete metrox. Then every subset of X is necessarily open. We can see that every set which consists of more than one element can be represented as a union of two or more disjoint singleton sets and is therefore disconnected. We call such a space* ***totally disconnected****.*

Chapter 7

Compactness

Definition 7.1. *Let (X, ρ) be a metric space and $K \subseteq X$ a subset. We say that K is **sequentially compact** if any sequence of elements of K has a subsequence which converges to a limit in K.*

Remark. *In light of the previous definition, it is obvious that K is compact in (X, ρ) if and only if it is compact in (X, σ) for any metric σ equivalent to ρ.*

Definition 7.2. *Let $\hat{S}$ be a family of subsets of a metric space (X, ρ) and $K \subseteq X$ a subset. We say that $\hat{S}$ is a **cover** of K if $K \subseteq \cup_{S \in \hat{S}} S$. If each member of $\hat{S}$ is open then we call it an **open cover** of K. Furthermore, if $\hat{S}$ is a cover of K and a subset $\hat{S}_0 \subseteq \hat{S}$ also covers K then $\hat{S}_0$ is called a **subcover** of $\hat{S}$. A cover (or subcover) is called **dinite** if it has a finite number of members.*

Definition 7.3. *Let (X, ρ) be a metric space and $K \subseteq X$ a subset. We say that K is **compact** if every open cover of K has a finite subcover.*

Theorem 7.4. *A subset of a metric space is compact if and only if it is sequentially compact.*

Definition 7.5. *Let (X, ρ) be a metric space and $K \subseteq X$ a subset. We say that K is **bounded** if for some $x \in X$ and $r > 0$ we have that $K \subseteq B_r(x)$.*

Theorem 7.6. *A compact set K of a metric space (X, ρ) is necessarily bounded and closed.*

Proof. Suppose that K is not bounded. Then for every $x \in X$, the family of open balls $B_n(x)$ for $n = 1, 2, \ldots$ form an open cover of K which does not have a finite subcover. This is a contradiction to the fact that K is compact. Now suppose that K is not closed. Then it does not contain at least one of its limit points. Consider a sequence of elements of K which converges to this limit point. Every subsequence of this sequence converges to the same limit point. Hence there exists a sequence of elements of K that does not have a subsequence which converges to a limit K. Hence K is not sequentially compact. But a set is sequentially compact if and only if it is compact which contradicts the fact that K is compact. $\square$

Lemma 7.7. *Let (X, ρ) be a metric space and $K \subseteq X$ a compact set. Any closed subset $B \subseteq K$ is necessarily compact.*

Proof. Let K be a compact set and $B \subseteq K$ a closed subset. Now any sequence $\{x_n\} \subseteq B$ must contain a convergent subsequence as $\{x_n\} \subseteq K$ which is a sequentially compact set. Since B is closed, it must contain all of its limit points. Hence $\{x_n\}$ has a subsequence which converges to a limit in B and B is thus sequentially compact and therefore compact. $\square$

Lemma 7.8. *Let (x, ρ) and (Y, σ) be metric spaces and $K \subseteq X$ and $L \subseteq L$ compact subsets. Then $K \times L \subseteq X \times Y$ with the metric*

$$d((x_1, y_1), (x_2, y_2)) = \sqrt{\rho(x_1, x_2)^2 + \sigma(y_1, y_2)^2}$$

is compact.

Proof. Let (x_n, y_n) be an arbitrary sequence in $K \times L$, we claim that it has a subsequence that converges to a limit $(x, y) \in K \times L$. Since K is compact, there is a subsequence x_{n_k} which coverges to a limit $x \in K$ as $k \to \infty$. Since L is compact, there sequence y_{n_k} has a subsequence $y_{n_{k_i}}$ which converges to a limit $y \in L$ as $i \to \infty$. Since $x_{n_k} \to x$ as $k \to \infty$ we also have that $x_{n_{k_i}} \to x$ as $i \to \infty$.

Now by the definition of convergence, we can reformulate the above as $\rho(x_{n_{k_i}}, x) \to 0$ and $\sigma(y_{n_{k_i}}, y) \to 0$ as $i \to \infty$. This implies that $d((x_{n_{k_i}}, y_{n_{k_i}}), (x, y)) \to 0$ as $i \to \infty$. In other words, $(x_{n_{k_i}}, y_{n_{k_i}}) \to (x, y) \in K \times L$. Therefore any sequence (x_n, y_n) of elements of $K \times L$ has a subsequence that converges to a limit in $K \times L$. Therefore $K \times L$ is sequentially compact and is hence compact. $\square$

Theorem 7.9. *A bounded and closed subset of* $\mathbb{R}^n$ *is necessarily compact.*

Proof. Since any bounded subset lies within a closed cube Q^n, we just have to show that the cube is compact and the compactness of the subset will follow from Lemma 7.7. Now if Q^1 and Q^{n-1} are both compact then by by Lemma 7.8, Q^n is also compact. Therefore it is sufficient to show that a closed interval of $\mathbb{R}$ is compact and we will then be able to apply induction.
Let $\{x_n\}$ be a sequence of elements of the closed interval $[a, b]$ where $b > a$. We begin by splitting the interval into two intervals of equal length $\frac{\delta}{2}$ where $\delta = b - a$. Obviously one (or even both) of the intervals contain infinitely many elements of $\{x_n\}$. Choose such an element from that interval and denote it y_1.
Now we again split the interval of length $\frac{\delta}{2}$ into two intervals of length $\frac{\delta}{4}$. Again one of these two intervals will contain infinitely many elements of $\{x_n\}$. We choose one of this elements, distinct from y_1, and denote it by y_2. Continuing like this, we obtain a sequence $\{y_k\}$ of the sequence $\{x_n\}$ such that each y_k lies in an interval of length $2^{-k_0}\delta$ for all $k \geq k_0$ for some constant k_0. It is clear that $\{y_k\}$ is a Cauchy sequence.
Now since $\mathbb{R}$ is a complete metric space, $\{y_k\}$ must converge to some limit. But $[a, b]$ is a closed set and must contain all of its limit points and hence $\{y_k\}$ converges to an element of $[a, b]$. We have shown that an arbitrary sequence $\{x_n\}$ has a convergent subsequence $\{y_n\}$ which converges to an element of $[a, b]$ and therefore $[a, b]$ must be sequentially compacy. It thus follows that $[a, b]$ is compact. □

Theorem 7.10. *Let* (X, ρ) *and* (Y, d) *be metric spaces and* $T : X \to Y$ *a continuous mapping. Then if* $K \subseteq X$ *is a compact set,* $T(K)$ *is necessarily compact.*

Proof. Let y_n be an arbitrary sequence of elements of $T(K)$. Then $y_n = Tx_n$ for some elements of a sequence $x_n \in K$. Since K is compact, it is also sequentially compact. We therefore have that $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ which converges to a limit $x \in K$. Now Theorem 4.2 implies that the sequence $y_{n_k} = Tx_{n_k}$ converges to the limit $Tx \in T(K)$. Hence $T(K)$ is sequentially compact and is thus compact. □

Definition 7.11. *A metric space* (X, ρ) *is said to be* ***compact*** *if the set* X *is itself compact.*

Theorem 7.12. *A compact metric space* (X, ρ) *is necessarily complete.*

Proof. Let $\{x_n\} \subseteq X$ be an arbitrary Cauchy sequence. Since X is compact, this sequence has a subsequence that converges to a limit in X. Now Lemma 5.3 implies that $\{x_n\}$ also converges to this limit. Hence any Cauchy sequence in X converges to a limit in X and X is thus complete. □

Theorem 7.13. *Let (X, ρ) and (Y, σ) be two compact metric spaces and $T : (X, \rho) \to (Y, \sigma)$ a continuous bijective mapping. Then the inverse mapping T^{-1} is also continuous.*

Proof. By Theorem 4.8, we know that T^{-1} is continuous if and only if $B \subseteq X$ closed $\implies T(B) \subseteq Y$ is also closed. If B is closed then by Lemma 7.7, it is necessarily compact as the subset of a compact space. The image of compact sets under continuous mappings are compact and therefore $T(B)$ is compact. It follows that $T(B)$ must be bounded and, more importantly, closed as required. □

Example 7.14. *Let X be the space of continuously differentiable functions on a closed interval $[a, b]$ and ρ and σ be metrics on X given by*

$$\rho(f, g) = \sup_{x\in[a,b]} |f(x) - g(x)| + \sup_{x\in[a,b]} |f'(x) - g'(x)|$$
$$\sigma(f, g) = \sup_{x\in[a,b]} |f(x) - g(x)|$$

The map $(X, \rho) \to (X, \sigma)$ is a bijection and is continuous because $f_n \xrightarrow{\rho} f \implies f_n \xrightarrow{\sigma} f$. However, the inverse mapping is not continuous. Consider the sequence $f_n(x) = n^{-1} \sin(n^2 x)$ converges to the zero function with respect to the metric σ but does not converge with respect to the metric ρ.

Definition 7.15. *We say that a function f defined on a metric space (X, ρ) is* ***uniformly continuous*** *if*

$$\forall \varepsilon > 0\, \exists \delta > 0 \text{ such that } \forall x, y, \rho(x, y) < \delta \implies |f(x) - f(y)| < \varepsilon$$

Remark. *Note that this definition is subtley different from that of ordinary continuity. Ordinary continuity guarantees the existence of a δ for any $x \in X$ but such a δ may be dependent on x. In other words, the δ depends on the distance between x and y. On the other hand, uniform continuity guarantees the existence of a single delta regardless of the given x and y.*

Theorem 7.16. *Let (X, ρ) be a compact metric space. Then any continuous function f on (X, ρ) is uniformly continuous.*

Proof. Fix $\varepsilon > 0$. Since f is a continuous function, we have that, by definition there exists a $\delta_x > 0$ such that

$$\rho(y, x) < \delta_x \implies |f(y) - f(x)| < \frac{\varepsilon}{2}$$

Now let $J_x = B_{\frac{\delta_x}{2}}(x)$ be an open ball. The collection on balls $\{J_x\}_{x \in X}$ obviously forms an open cover of X. Now since X is compact, such an open cover must contain a finite subcover. That is to say that there exists a finite collection of points $x_1, \ldots, x_k$ such that $\cup_{n=1}^{k} J_{x_n}$. Now define $\delta = \min\{\delta_{x_1}, \ldots, \delta_{x_k}\}$.
Now let $x, y \in X$ and $\rho(x, y) < \delta$. Since $X = \cup_{n=1}^{k}$, we can always find an n such that $x \in J_{x_n}$. In other words, $\rho(x, x_n) < \frac{\delta_{x_n}}{2}$. Now the triangle inequality implies that

$$\rho(y, x_n) \leq \rho(x, x_n) + \rho(x, y) < \frac{\delta_{x_n}}{2} + \delta \leq \delta_{x_n}$$

Hence we see that

$$|f(y) - f(x)| \leq |f(y) - f(x_n)| + |f(x_n) - f(x)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Therefore we have shown that $\forall \varepsilon > 0 \, \exists \, \delta > 0$ such that $\forall x, y, \rho(x, y) < \delta \implies |f(x) - f(y)| < \varepsilon$. □

Definition 7.17. *Let (K, ρ) be a metric space. Then $C(K)$ denotes the linear space of continuous functions $f : K \to \mathbb{R}$ equipped with the norm $||f|| = \sup_{x \in K} |f(x)|$.*

Theorem 7.18. *(Weierstrass' Approximation Theorem)*
Let $\mathcal{P}$ represent the set of polynomials in $C[a, b]$ for some closed bounded interval $[a, b]$. Then $\overline{\mathcal{P}} = C[a, b]$.

Proof. The proof follows from the Stone-Weierstrass theorem below. □

Remark. *The previous theorem is equivalent to the fact that any continuous function on a closed bounded interval can be uniformly approximated by polynomials.*

Definition 7.19. *Let X be a linear space and let $x, y \in X$. We say that X is an **algebra** if $xy \in X$.*

Example 7.20. *$C(K)$ is an algebra.*

Definition 7.21. *Let X be an algebra and $A \subseteq X$. We say that A is a* ***subalgebra*** *of X if A is a linear space and if $x, y \in A$ then $xy \in A$.*

Definition 7.22. *Let (X, ρ) be a metric space and $A \subseteq X$. We say that A is* ***dense*** *in X if $\overline{A} = X$.*

Theorem 7.23. *(Stone-Weierstrass)*
Let K be a compact metric space and $\mathcal{P}$ a subalgebra of $C(K)$. If

1. *$\mathcal{P}$ contains the constant functions*
2. *for all $x, y \in K$, there exists a function $f \in \mathcal{P}$ such that $f(x) \neq f(y)$*

then $\overline{P} = C(K)$. In other words, $\overline{P}$ is dense in $C(K)$.

Example 7.24. *Consider finite sums of the form*

$$c + \sum_{n=1}^{k} a_n \sin(nx) + \sum_{j=1}^{l} b_j cos(jx)$$

for some constants c, a_n, b_j. Such sums are referred to as trigonometric polynomials. We shall show that the Stone-Weierstrass theorem implies that the trigonometric polynomials are dense in $C[a, b]$ for any closed bounded interval $[a, b]$ provided that $b - a < 2\pi$.
The product formulas for trigonometric functions confirm that the set of trigonometric polynomials is a subalgebra of $\mathcal{P}$.
Now, if $\sin(nx) = \sin(ny)$ and $\cos(nx) = \cos(ny)$ for all n, the product formulas imply the following

$$0 = \sin(nx) - \sin(ny) = 2\cos\frac{n(x+y)}{2}\sin\frac{n(x-y)}{2}$$
$$0 = \cos(ny) - \cos(nx) = 2\sin\frac{n(x+y)}{2}\sin\frac{n(x-y)}{2}$$

It follows that $\sin\frac{n(x-y)}{2} = 0$ for all n. This is only possible if $\frac{x-y}{2} = k\pi$ for some integer k. But if $b - a < 2\pi$ then $x = y$ and therefore the trigonometric polynomials are dense in $C[a, b]$.

Chapter 8

Integration

8.1 Step functions

Definition 8.1. *Let ψ be a complex valued function on a bounded interval $[a, b]$. We say that ψ is a* ***step function*** *if there exists a finite collection of intervals $I_k \subseteq [a, b]$ with $k = 1, 2, \ldots, N$ such that*

1. $\cup_{k=1} I_k = [a, b]$
2. $I_j \cap I_k = \varnothing$ *for* $j \neq k$
3. ψ *is constant on each individual* I_k

Remark. *Any step function ψ is determined by a collection of intervals I_k and constants $c_k = \psi|_{I_k}$. We write $\psi \sim \{ I_k, c_k \}$ if ψ is constant on the intervals on I_k and takes the value c_k on the interval I_k.*

Lemma 8.2. *The set of all step functions form a linear space. In other words, let ψ_1, ψ_2 be step functions and $\lambda \in \mathbb{C}$. Then*

1. $\lambda\psi_1$ *is a step function*
2. $\psi_1 + \psi_2$ *is a step function*

Proof.

Part 1: Let $\psi \sim \{ I_k, c_k \}$. Then $\lambda\psi \sim \{ I_k, \lambda c_k \}$. Therefore $\lambda\psi$ is a step function.

Part 2: Now let $\psi_1 \sim \{ I_j, c_j \}$ for $j = 1, \ldots, N_1$ and $\psi_2 \sim \{ \tilde{I}_k, \tilde{c_k} \}$ for $k = 1, \ldots, N_2$. The intersections $I_j \cap \tilde{I}_k$ are obviously all disjoint intervals and

$I_j \cap \tilde{I}_k \subseteq [a,b]$. By definition, we have that $\cup_{j=1}^{N_1} I_j = [a,b]$ and $\cup_{k=1}^{N_2} \tilde{I}_k = [a,b]$. Therefore, we have that

$$\bigcup_{j=1}^{N_1} \bigcup_{k=1}^{N_2} \left(I_j \bigcap \tilde{I}_k \right) = [a,b]$$

Hence the collection of intervals $\{ I_j \cap \tilde{I}_k \}$ for $j = 1, \ldots, N_1$ and $k = 1, \ldots, N_2$ satisfies the first two conditions of the definition of a step function. Furthermore, the function $\psi_1 + \psi_2$ is constant on each interval $I_j \cap \tilde{I}_k$ and takes the value $c_j + \tilde{c}_k$. Hence $\psi_1 + \psi_2$ is a step function and $\psi_1 + \psi_2 \sim \{ I_j \cap \tilde{I}_k, c_j + \tilde{c}_k \}$. □

Definition 8.3. *Consider the step functions as a subset X of $B[a,b]$, the set of bounded functions on a closed bounded interval $[a,b$. Then the set $R[a,b] = \overline{X}$ is called the set of* ***Riemann integrable*** *functions and consists of all step functions and functions f that can be approximated by step functions.*

Remark. *By Corollary 3.19, $f \in R[a,b]$ if and only if there exists a sequence of step functions ψ_n converging to f in $B[a,b]$.*

Theorem 8.4. *The continuous functions $C[a,b]$ are a subset of the Riemann integrable functions $R[a,b]$.*

Proof. Let $f \in C[a,b]$ be a continuous function. Then, since the interval $[a,b]$ is comapct, f is uniformly continuous (by Theorem 7.16). By definition, this means that

$$\forall \varepsilon > 0, \exists \delta > 0 \text{ such that } \forall x, y, |x-y| \leq \delta \implies |f(x) - f(y)| \leq \varepsilon$$

Therefore we can choose a δ_n such that $|f(x) - f(y)| \leq \frac{1}{n}$ whenever $|x-y| \leq \delta_n$. We now split the interval $[a,b]$ into the union of pairwise disjoint intervals I_k of lengths not greater than δ_n. Now choose an arbitrary point $x_k \in I_k$ for each k and let $c_k = f(x_k)$. Denote $\psi \sim \{ I_k, c_k \}$ to be the corresponding step function.

Now choose an element $x \in [a,b]$. Then $x \in I_k$ for some k and, therefore, $|x - x_k| \leq \delta_n$. Furthermore,

$$|f(x) - \psi_n(x)| = |f(x) - c_k| = |f(x) - f(x_k)| \leq \frac{1}{n}$$

Thus $|f(x) - \psi_n(x)| \leq \frac{1}{n}$ for all $x \in [a, b]$. This implies that

$$\sup_{x \in [a,b]} |f(x) - \psi_n(x)| \leq \frac{1}{n}$$

Now, as $n \to \infty$, the above goes to 0 and thus $f \in R[a, b]$. □

Definition 8.5. *Let f be a complex-valued function on a closed, bounded interval $[a, b]$. We say that f is **piecewise continuous** if there exists a finite collection of intervals $I_k \subseteq [a, b]$ for $k = 1, \ldots, N$ such that*

1. $\cup_{k=1}^{N} I_k = [a, b]$
2. $I_j \cap I_k = \varnothing$ *for* $j \neq k$
3. *f is continuous inside each interval I_k and has a finite limit at the end points of the interval I_k*

Corollary 8.6. *Let f be a piecewise continuous function. Then $f \in R[a, b]$.*

Proof. By the previous Theorem, we can always find, given an interval I_k, a sequence of step functions $\psi_n^(k)$ such that

$$\sup_{x \in I_k} |f(x) - \psi_n^{(k)}(x)| \to 0 \tag{8.1}$$

as $n \to \infty$. We now extend $\psi_n^{(k)}$ by zero to the whole interval $[a, b]$ and define

$$\psi_n(x) = \sum_{k=1}^{N} \psi_n^{(k)}(x)$$

Then ψ_n is a step function defined on the interval $[a, b]$ and because of (8.1), we have that

$$\sup_{x \in [a,b]} |f(x) - \psi_n(x)| \to 0$$

as $n \to \infty$. Hence $f \in R[a, b]$. □

Definition 8.7. *Let ψ be a step function and $\psi \sim \{ I_k, c_k \}$ for $k = 1, \ldots, N$. Then we define the **integral** of $\psi(x)$ with respect to x between a and b to be*

$$\int_a^b \psi(x)\, dx = \sum_{k=1}^{N} c_k \mu(I_k)$$

where $\mu(I_k)$ is the length of the interval I_k.

Lemma 8.8. *Let ψ_1, ψ_2 be step functions and $\lambda \in \mathbb{C}$ a constant. Then*

$$\int_a^b \lambda\psi_1(x)\,dx = \lambda \int_a^b \psi_1(x)\,dx \tag{8.2}$$

$$\int_a^b \psi_1(x) + \psi_2(x)\,dx = \int_a^b \psi_1(x)\,dx + \int_a^b \psi_2(x)\,dx \tag{8.3}$$

Proof.

Part 1: This follows immediately from the definition of the integral and the fact that $\lambda\psi_1 \sim \{ I_k, \lambda c_k \}$.

Part 2: Let $\psi_1 \sim \{ I_j, c_j \}$ for $n = 1, \ldots, N_1$ and $\psi_2 \sim \{ \tilde{I}_k, \tilde{c}_k \}$ for $k = 1, \ldots, N_2$. Then

$$\psi_1 + \psi_2 \sim \{ I_j \cap \tilde{I}_k, c_j + \tilde{c}_k \}$$

for $j = 1, \ldots, N_1$ and $k = 1, \ldots, N_2$. Therefore

$$\begin{aligned}
\int_a^b \psi_1(x) + \psi_2(x)\,dx &= \sum_{j=1}^{N_1} \sum_{k=1}^{N_2} c_j + \tilde{c}_k \mu(I_j \cap \tilde{I}_k) \\
&= \sum_{j=1}^{N_1} \left(c_j \sum_{k=1}^{N_2} \mu(I_j \cap \tilde{I}_k) \right) + \sum_{k=1}^{N_2} \left(\tilde{c}_k \sum_{j=1}^{N_1} \mu(I_j \cap \tilde{I}_k) \right) \\
&= \sum_{j=1}^{N_1} c_j \mu(I_j) + \sum_{k=1}^{N_2} \tilde{c}_k \mu(\tilde{I}_k) \\
&= \int_a^b \psi_1(x)\,dx + \int_a^b \psi_2(x)\,dx
\end{aligned}$$

□

Lemma 8.9. *Let ψ be a step function. Then*

$$\left| \int_a^b \psi(x)\,dx \right| \leq (b-a) \sup_{x \in [a,b]} |\psi(x)|$$

Proof. We have that

$$\begin{aligned}\left|\int_a^b \psi(x)\,dx\right| &= \left|\sum_{k=1}^{N} c_k\mu(I_k)\right| \\ &\leq \sum_{k=1}^{N} |c_k|\mu(I_k) \\ &\leq \sup_{x\in[a,b]} |\psi(x)| \sum_{k=1}^{N} \mu(I_k) \\ &= (b-a) \sup_{x\in[a,b]} |\psi(x)|\end{aligned}$$

□

8.2 Definition and Basic Properties of Integrals

Definition 8.10. *Let $f \in R[a,b]$ be a Riemann integrable function and ψ_n such that $n \in \mathbb{N}$ a sequence of step functions convergeing to f in $B[a,b]$. We define the* ***integral*** *of $f(x)$ with respect to dx between a and b by*

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \int_a^b \psi_n(x)\,dx$$

Furthermore, we define $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$.

Remark. *This definition only makes sense if the limit actually exists and is independent of the choice of sequence ψ_n. Assume that $\psi_n \to f$ in $B[a,b]$. Then ψ_n is a Cauchy sequence in $B[a,b]$ and, by definition,*

$$\sup_{x\in[a,b]} |\psi_(x) - \psi_m(x)| \to 0$$

as $m,n \to \infty$. Therefore, by Lemma 8.8 and Lemma 8.9, we have that

$$\begin{aligned}\left|\int_a^b \psi_n(x)\,dx - \int_a^b \psi_m(x)\,dx\right| &= \left|\int_a^b (\psi_n(x) - \psi_m(x))\,dx\right| \\ &\leq (b-a) \sup_{x\in[a,b]} |\psi_n(x) - \psi_m(x)| \to 0\end{aligned}$$

as $m, m \to \infty$. Hence the sequence $\int_a^b \psi_n(x)$ is Cauchy and the limit must exist.
Now let $\tilde{\psi}_n$ be another sequence of step functions converging to f in $B[a,b]$. Then

$$\sup_{x\in[a,b]} |\psi_n(x) - \tilde{\psi}_n(x)| \leq \sup_{x\in[a,b]} |\psi_n(x) - f(x)| + \sup_{x\in[a,b]} |f(x) - \tilde{\psi}_n(x)| \to 0$$

as $n \to \infty$. Hence

$$\begin{aligned} \left|\int_a^b \psi_n(x)\ dx - \int_a^b \tilde{\psi}_n(x)\ dx\right| &= \left|\int_a^b \Big(\psi_n(x) - \tilde{\psi}_n(x)\Big)\ dx\right| \\ &\leq (b-a) \sup_{x\in[a,b]} |\psi_n(x) - \tilde{\psi}_n(x)| \end{aligned}$$

as $n \to \infty$. Hence the limit does not depend on the choice of sequence ψ_n.

Lemma 8.11. *Let $f \in R[a,b]$. Then*

$$\left|\int_a^b f(x)\ dx\right| \leq (b-a) \sup_{x\in[a,b]} |f(x)|$$

Proof. Let ψ_n be a sequence of step functions that converge to f. By Lemma 8.9 and the triangle equality, it follows that

$$\begin{aligned} \left|\int_a^b \psi_n(x)\ dx\right| &\leq (b-a) \sup_{x\in[a,b]} |\psi_n(x)| \\ &= (b-a) \sup_{x\in[a,b]} |f(x)| + (b-a) \sup_{x\in[a,b]} |\psi_n(x) - f(x)| \end{aligned}$$

Therefore

$$\begin{aligned} \left|\int_a^b f(x)\ dx\right| &= \left|\lim_{n\to\infty} \int_a^b \psi_n(x)\ dx\right| \\ &= \lim_{n\to\infty} \left|\int_a^b \psi_n(x)\ dx\right| \\ &\leq (b-a) \sup_{x\in[a,b]} |f(x)| + (b-a) \lim_{n\to\infty} \sup_{x\in[a,b]} |\psi_n(x) - f(x)| \\ &= (b-a) \sup_{x\in[a,b]} |f(x)| \end{aligned}$$

□

Definition 8.12. *Let X be a linear space. A linear map $F : X \to \mathbb{C}$ is said to be a* ***linear functional*** *on X.*

Theorem 8.13. *Let $f \in R[a,b]$ be a Riemann integrable function. Then the map*

$$f \to \int_a^b f(x)\ dx$$

is a linear uniformly continuous functional on $R[a,b]$.

Proof. We first show that the functional is linear. Let $f, g \in R[a,b]$ and $\lambda, \mu \in \mathbb{C}$. Let ψ_n and ϕ_n such that $n \in \mathbb{N}$ be a sequence of step functions that converge to f and g respectively. Then by Lemma 8.8, we have that

$$\int_a^b (\lambda\psi_n(x) + \mu\phi_n(x))\ dx = \lambda \int_a^b \psi_n(x)\ dx + \mu \int_a^b \phi_n(x)\ dx$$

Now, taking the limit as $n \to \infty$ on both sides of the above equation, we see that the map must be linear.
We now show that the functional is uniformly continuous. In order to do this, we need to show that $\forall\, \varepsilon > 0, \exists\, \delta > 0$ such that whenever $\sup_{x\in[a,b]} |f(x) - g(x)| < \delta$ then $|\int_a^b f(x)\ dx - \int_a^b g(x)\ dx| < \varepsilon$. Lemma 8.11 and the linearity of the functional imply that

$$\begin{aligned}\left|\int_a^b f(x)\ dx - \int_a^b g(x)\ dx\right| &= \left|\int_a^b (f(x) - g(x))\ dx\right| \\ &\le (b-a) \sup_{x\in[a,b]} |f(x) - g(x)|\end{aligned}$$

Now fix $\varepsilon > 0$. It is easy to see that for $\delta = (b-a)^{-1}\varepsilon$, whenever $\sup_{x\in[a,b]} |f(x) - g(x)| < \delta$ then $|\int_a^b f(x)\ dx - \int_a^b g(x)\ dx| < \varepsilon$. □

Definition 8.14. *Let $f : [a,b] \to \mathbb{R}$ be a function. We denote*

$$f_+(x) = \begin{cases} f(x) & \text{if } f(x) \ge 0 \\ 0 & \text{if } f(x) < 0 \end{cases}, \quad f_-(x) = \begin{cases} 0 & \text{if } f(x) \ge 0 \\ -f(x) & \text{if } f(x) < 0 \end{cases}$$

Remark. *Obviously, f_+ and f_- are non-negative functions on $[a,b]$ and $f(x) = f_+(x) - f_-(x), |f(x)| = f_+(x) + f_-(x)$ and $f_+(x)f_-(x) = 0$. Hence any real-valued function can be represented as a linear combination of non-negative functions. Any complex calued function f is a linear combination of real valued functions $Re(f)$ and $Im(f)$ and therefore it can also be represented as a linear combination of non-negative functions.*

Proposition 8.15. *Let $f \in R[a,b]$ be a non-negative Riemann integrable function. Then $\int_a^b f(x)\,dx \geq 0$.*

Proof. Let ψ_n such that $n \in \mathbb{N}$ be a sequence of step functions that converges to f. Now,

$$|f(x) - (\psi_n)_+(x)| \leq |f(x) - \psi_n(x)|$$
$$\implies \sup_{x\in[a,b]} |f(x) - (\psi_n)_+(x)| \leq \sup_{x\in[a,b]} |f(x) - \psi_n(x)|$$

for all $x \in [a,b]$. Therefore, $(\psi_n)_+$ also converges to f. Obviously, $\int_a^b (\psi_n)_+(x)\,dx \geq 0$ for all n. Passing to the limit $n \to \infty$, we see that $\int_a^b f(x)\,dx \geq 0$. □

Proposition 8.16. *Let $f \in R[a,b]$ be a Riemann integrable function. Then $|f| \in R[a,b]$ and*

$$\left|\int_a^b f(x)\,dx\right| \leq \int_a^b |f(x)|\,dx$$

Proof. Let ψ_n such that $n \in \mathbb{N}$ be a sequence of step functions convergent to f in $B[a,b]$. Then obviously, the sequence $|\psi_n|$ converges to $|f|$ in $B[a,b]$. Therefore $|f| \in R[a,b]$.
The inequality holds for any step function ψ_n and thus, passing to the limit $n \to \infty$, we obtain the desired result. □

Proposition 8.17. *Let $f \in R[a,b]$ be a Riemann integrable function and $c \in [a,b]$. Then*

$$\int_b^a f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$

Proof. It suffices to show the equality for step functions, after which the general result follows by passing to the limit $n \to \infty$.
Let ψ be a step function and $\psi \sim \{ I_k, c_k \}$. Let $c \in I_{k_0}$ for some k_0. Now denote $I'_{k_0} = \{ x \in I_{k_0} \mid x < c \}$ and $I''_{k_0} = \{ x \in I_{k_0} \mid x \geq c \}$. Then

$$\begin{aligned}\int_a^b \psi(x)\,dx &= \sum_k c_k\mu(I_k)\\ &= \sum_{k<k_0} c_k\mu(I_k) + c_{k_0}\mu(I_{k_0}) + \sum_{k>k_0} c_k\mu(I_k)\\ &= \int_a^c \psi(x)\,dx + \int_c^b \psi(x)\,dx\end{aligned}$$

□

8.3 Unbounded Functions and Unbounded Intervals

Definition 8.18. *Let $c \in (a,b)$. If $f \in R[a, c-\varepsilon]$ and $f \in R[c+\delta, b]$ for all positive ϵ and δ then we define*

$$\int_a^b f(x)\,dx = \lim_{\varepsilon\to 0}\int_a^{c-\varepsilon} f(x)\,dx + \lim_{\delta\to 0}\int_{c+\delta}^b f(x)\,dx$$

If either of the limits on the right hand side does not exist (or is $\pm\infty$) then we say that $\int_a^b f(x)\,dx$ is not defined.

Example 8.19. *Let $f(x) = x^{-1}$ and $a < 0 < b$. Then*

$$\int_a^{-\varepsilon} f(x)\,dx = \log\varepsilon - \log|a| \to -\infty, \varepsilon \to 0$$
$$\int_\delta^b f(x)\,dx = \log b - \log\delta \to \infty, \delta \to 0$$

hence the integral $\int_a^b f(x)$ is undefined. On the other hand,

$$\lim_{\varepsilon\to 0}\left(\int_a^{-\varepsilon} f(x)\,dx + \int_\varepsilon^b f(x)\,dx\right) = \lim_{\varepsilon\to 0}(\log b - \log|a|) = \log b - \log|a| \tag{8.4}$$

Hence it is important that we consider seperate limits in the definition of such an integral. The integral defined by Definition 8.18 is often referred to as an ***improper integral*** *while the integral in Equation (8.4) is referred to as a* ***singular integral****.*

Definition 8.20. *Let $f \in R[a,b]$ be a Riemann integrable function. Then for all $-\infty < a < b < \infty$, we define*

$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx$$
$$\int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty}\int_a^b f(x)\,dx$$
$$\int_{-\infty}^\infty f(x)\,dx = \lim_{a\to-\infty}\int_{-\infty}^a f(x)\,dx + \lim_{a\to-\infty}\int_a^\infty f(x)\,dx$$

Remark. *Let (a,b) be an unbounded interval and $R(a,b)$ the set of Riemann integrable functions on (a,b). Then the map $f \to \int_a^b f(x)\ dx$ is not a continuous functional on $R(a,b)$ with respect to the standard metric $\rho(f,g) = \sup_{x\in(a,b)} |f(x) - g(x)|$. Indeed, consider the sequence of functions*

$$f_n(x) = \begin{cases} \frac{1}{n} & \text{if } -n \leq x \leq n \\ 0 & \text{if otherwise} \end{cases}$$

Such a sequence obviously converges to the zero function on $B(-\infty,\infty)$ but $\int_{-\infty}^{\infty} f_n(x)\ dx = 2$ for all n.

8.4 Integrals depending on a parameter

Theorem 8.21. *Let $[a,b]$ and $[\alpha,\beta]$ be bounded closed intervals and $f(x,t)$ a continuous function on $[a,b]\times[\alpha,\beta]$. Then $\int_a^b f(x,t)\ dx$ is a continuous function on $[\alpha,\beta]$.*

Proof. The set $[a,b]\times[\alpha,\beta]$ is compact. Hence f is uniformly continuous on $[a,b]\times[\alpha,\beta]$. By definition, we have that

$$\forall\,\varepsilon > 0, \exists\,\delta > 0 \text{ such that } |t - t_0| < \delta \implies |f(x,t) - f(x,t_0)| < \varepsilon$$

This means that

$$\sup_{x\in[a,b]} |f(x,t) - f(x,t_0)| \to 0$$

as $t \to t_0$. Therefore

$$\begin{aligned} \left| \int_a^b f(x,t)\ dx - \int_a^b f(x,t_0)\ dx \right| &= \left| \int_a^b (f(x,t) - f(x,t_0)) \right| \\ &\leq (b-a) \sup_{x\in[a,b]} |f(x,t) - f(x,t_0)| \to 0 \end{aligned}$$

as $t \to t_0$ as required. □

Theorem 8.22. *Let $[a,b]$ and $[\alpha,\beta]$ be two closed bounded intervals and $f(x,t)$ a continuous function on $[a,b]\times[\alpha,\beta]$. Furthermore, assume that f is continuously differentiable in t. Then*

$$\frac{d}{dt}\left(\int_a^b f(x,t)\ dx\right) = \int_a^b \frac{\partial}{\partial t} f(x,t)\ dx$$

Proof. We have that

$$\frac{d}{dt}\left(\int_a^b f(x,t)\,dx\right) = \lim_{\delta\to 0}\delta^{-1}\left(\int_a^b f(x,t+\delta) - \int_a^b f(x,t)\,dx\right)$$

Now applying the mean value theorem, we see that

$$\begin{aligned} fracddt\left(\int_a^b f(x,t)\,dx\right) &= \lim_{\delta\to 0}\delta^{-1}\left(\int_a^b f(x,t+\delta) - \int_a^b f(x,t)\,dx\right) \\ &= \lim_{\delta\to 0}\int_a^b \delta^{-1}[f(x,t+\delta) - \int_a^b f(x,t)\,dx] \\ &= \lim_{\delta\to 0}\int_a^b \frac{\partial f}{\partial t}(x,t+\delta^*)\,dx \end{aligned}$$

where $0 \le \delta^* \le \delta$. Obviously, if $\delta \to 0$ then $\delta^* \to 0$. Now by Theorem 8.21, the integral is a continuous function and hence

$$\begin{aligned} \frac{d}{dt}\left(\int_a^b f(x,t)\,dx\right) &= \lim_{\delta^*\to 0}\int_a^b \frac{\partial f}{\partial t}(x,t+\delta^*)\,dx \\ &= \int_a^b \frac{\partial f}{\partial t}(x,t)\,dx \end{aligned}$$

□

Theorem 8.23. *Let $[a,b]$ be a closed bounded interval and $f_n \in R[a,b]$ a sequence of Riemann integrable functions converging to a function $f \in R[a,b]$. In other words,*

$$\sup_{x\in[a,b]}\left|f(x) - \sum_{n=1}^{k} f_n(x)\right| \overset{k\to\infty}{\to} 0$$

then

$$\sum_{n=1}^{\infty}\left(\int_a^b f_n(x)\,dx\right) = \int_a^b f(x)\,dx$$

Proof. Since the integral is a linear continuous functional on $R[a,b]$, we have that

$$\int_a^b f(x)\,dx = \int_a^b \left(\lim_{k\to\infty}\sum_{n=1}^{k} f_n(x)\right)\,dx$$
$$= \lim_{k\to\infty}\sum_{n=1}^{k}\left(\int_a^b f_n(x)\,dx\right)$$
$$= \sum_{n=1}^{\infty}\left(\int_a^b f_n(x)\,dx\right)$$

□

Remark. *This theorem essentially states that we can integrate a convergent series term by term.*

Corollary 8.24. *Let $[a,b]$ be a bounded interval and f_n a sequence of continuously differentiable functions on $[a,b]$ such that*

1. $\sum_{n=1}^{\infty} f_n(x) = f(x)$ *for all* $x \in [a,b]$
2. *the series* $\sum_{n=1}^{\infty} f_n'$ *is uniformly convergent*

then the function f is continuously differentiable and

$$\sum_{n=1}^{\infty} f_n'(x) = f'(x)$$

for all $x \in [a,b]$.

Proof. Let $\sum_{n=1}^{\infty} = \tilde{f}$. Since the series $\sum n = 1^{\infty}$ converges uniformly in $C[a,b]$, the function $\tilde{f}$ is continuous. Now, Theorem 8.23 and the fundamental theorem of calculus imply that

$$\int_a^x \tilde{f}(t)\,dt = \sum_{n=1}^{\infty}\int_a^x f_n'(t)\,dt = \sum_{n=1}^{\infty}(f_n(x) - f_n(a)) = f(x) - f(a)$$

for all $x \in [a,b]$. Now, the fundamental theorem of calculus implies that f is continuously differentiable and $f' = \tilde{f}$. □

8.5 Picard's Existence Theorem for First Order Differential Equations

Let f be a real valued function defined on an open domain $\Omega \subseteq \mathbb{R}^2$. Consider the ordinary differential equation

$$\frac{d\varphi}{dx} = f(x, \varphi(x))$$

with the initial condition $\varphi(x_0) = \varphi_0$ where x is a one dimensional variable and φ is a function of x and φ_0 is some constant.

Theorem 8.25. *(Picard's Theorem)*
Let $(x_0, \varphi_0) \subseteq \Omega$ and f a continuous function satisfying the Lipschitz condition

$$|f(x, y_1) - f(x, y_2)| \leq c|y_1 - y_2|$$

where c is some constant. Then the ordinary differential equation above has a unique solution on some interval $[x_0 - \delta, x_0 + \delta]$.

Proof. Since f is a continuous function, we have that $|f(x, y)| \leq R$, for some constant R whenever (x, y) lie in a sufficiently small ball B around the point (x_0, ϕ_0). Now we choose a small positive constant δ such that $(x, y) \in B$ whenever $|x - x_0| \leq \delta$ and $|y - \varphi_0| \leq R\delta$. Furthermore, we require that $c\delta < 1$.
We denote by C^* the closed ball of radius $R\delta$ centered at φ_0 in the space $C[x_0 - \delta, x_0 + \delta]$. In other words, C^* is the set of all continuous functions ϕ on the interval $[x_0 - \delta, x_0 + \delta]$ such that

$$\sup_{x \in [x_0 - \delta, x_0 + \delta]} |\phi(x) - \varphi_0| \leq R\delta$$

By Theorem 5.6, C^* with the standard metric is complete.
By the fundamental theorem of calculus, with the initial condition $\varphi(x_0) = \varphi_0$ is equivalent to the integral equation

$$\varphi(x) = \varphi_0 + \int_{x_0}^{x} f(t, \varphi(t))\, dt \tag{8.5}$$

Now consider the map $T : C^* \to C[x_0 - \delta, x_0 + \delta]$ given by

$$T\psi(x) = \psi_0 + \int_{x_0}^{x} f(t, \psi(t))\, dt$$

for $x \in [x_0 - \delta, x_0 + \delta]$. Then Equation (8.5) is equivalent to $T\varphi = \varphi$. Now, if $\psi \in C^*$ then we have that

$$|T\psi(x) - \varphi_0| \leq \left| \int_{x_0}^{x} f(t, \psi(t))\, dt \right| \leq R\delta$$

This implies that $T : C^* \to C^*$. Now,

$$\begin{aligned} |T\psi_1(x) - T\psi_2(x)| &\leq \left| \int_{x_0}^{x} |f(t, \psi_1(t)) - f(t, \psi_2(t))|\, dt \right| \\ &\leq c\delta \sup_{t \in [x_0 - \delta, x_0 + \delta]} |\psi_1(t) - \psi_2(t)| \end{aligned}$$

□